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Path Integral for Space-time Noncommutative Field Theory

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Abstract

The path integral for space-time noncommutative theory is formulated by means of Schwinger's action principle which is based on the equations of motion and a suitable ansatz of asymptotic conditions. The resulting path integral has essentially the same physical basis as the Yang-Feldman formulation. It is first shown that higher derivative theories are neatly dealt with by the path integral formulation, and the underlying canonical structure is recovered by the Bjorken-Johnson-Low (BJL) prescription from correlation functions defined by the path integral. A simple theory which is non-local in time is then analyzed for an illustration of the complications related to quantization, unitarity and positive energy conditions. From the view point of BJL prescription, the naive quantization in the interaction picture is justified for space-time noncommutative theory but not for the simple theory non-local in time. We finally show that the perturbative unitarity and the positive energy condition, in the sense that only the positive energy flows in the positive time direction for any fixed time-slice in space-time, are not simultaneously satisfied for space-time noncommutative theory by the known methods of quantization.

1 Introduction

The field theory in noncommutative space and time has a long history[1, 2, 3, 4]. The quantization of space-time noncommutative theory, which contains noncommutative parameters in the time direction, is known to be problematic; the difficulties are common to those general theories non-local in time. The past analyses of non-local theories are found, for example, in [5, 6, 7] and references quoted therein. First of all, no canonical formulation of such theories is known since a sensible definition of canonical momenta is not known. Naturally, several authors showed the violation of unitarity in space-time noncommutative theory[8, 9, 10]. On the other hand, it has been pointed out that a suitable definition of time-ordering operation restores the unitarity in space-time noncommutative theory[11, 12, 13]. In view of the conflicting statements in the literature, one may ask what is the sensible definition of quantized space-time noncommutative theory, in particular, if the naive quantization of space-time noncommutative theory in the interaction picture is really justified. One may also ask if the time ordering can be freely modified

without introducing other complications in space-time noncommutative theory. The main purpose of the present paper is to analyze these basic issues.

To analyze the properties of quantized theory whose canonical quantization is not known, one needs to define a quantized theory in a more general setting. As one of such possibilities it is shown that the path integral on the basis of Schwinger's action principle, which is based on the formally quantized equations of motion and a suitable ansatz of asymptotic conditions, provides a proper starting point of analyses. The validity of this approach is similar to that of the Yang-Feldman formulation[14] which has been utilized in quantizing the noncommutative theory, but the time ordering operation is more rigidly specified in the path integral. In this path integral approach, the canonical structure is recovered later by means of Bjorken-Johnson-Low (BJL) prescription[15] once one defines correlation functions by the path integral.

We first illustrate that we can provide a reliable basis for the quantization of higher derivative theory by the path integral described above, which may be regarded as a first step toward the quantization of space-time noncommutative theory. We show how to recover the canonical structure for higher derivative theory from the path integral formulation. We then discuss the quantization of a simple field theory non-local in time. Some of the basic issues related to the quantization itself and the unitarity and positive energy conditions are analyzed. In the framework of BJL prescription, it is shown that the quantization on the basis of a naive interaction picture is not justified if the interaction contains non-local terms in time. The path integral quantization breaks perturbative unitarity, but it ensures the positive energy condition in the sense that only the positive energy flows in the positive time direction for any fixed time-slice in space-time by means of Feynman's $m^2 - i\epsilon$ prescription. One can define a unitary S-matrix by using a modified time ordering, but the positive energy condition is spoiled together with a smooth Wick rotation to Euclidean theory in the modified time ordering.

We finally analyze the quantum theory of space-time noncommutative theory. In this theory it is shown that the naive quantization in the interaction picture is justified even after one incorporates the higher order corrections perturbatively in contrast to the naive theory non-local in time, though this does not provide a basis for the non-perturbative definition of quantization. The path integral quantization with the Feynman's $m^2 - i\epsilon$ prescription spoils the perturbative unitarity though the positive energy condition in the sense that only the positive energy flows in the positive time direction for any fixed time-slice in space-time is ensured. One can define a unitary S-matrix for space-time noncommutative theory by using a modified time ordring but the positive energy condition is spoiled together with a smooth Wick rotation to Euclidean theory.

2 Higher derivative theory and canonical structure

In this section we give a path integral formulation of higher derivative theory and then show how to recover the canonical structure from the path integral. This analysis is useful to understand the basis of path integrals defined by means of Schwinger's action principle.

For simplicity, we first study the theory defined by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) + \lambda\frac{1}{2}\phi(x)\Box^2\phi(x) \tag{2.1}$$

where[1]

$$\Box = \partial_\mu\partial^\mu \tag{2.2}$$

and λ is a real constant. The canonical formulation of higher derivative theory such as the present one has been analyzed in [16], for example.

We instead start with Schwinger's action principle and consider the Lagrangian with a source function $J(x)$

$$\mathcal{L}_J = \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) + \lambda\frac{1}{2}\phi(x)\Box^2\phi(x) + J(x)\phi(x). \tag{2.3}$$

The Schwinger's action principle starts with the equation of motion

$$\begin{aligned}&\langle+\infty| - \Box\hat{\phi}(x) + \lambda\Box^2\hat{\phi}(x) + J(x)| - \infty\rangle_J \\ &= \{-\Box\frac{\delta}{i\delta J(x)} + \lambda\Box^2\frac{\delta}{i\delta J(x)} + J(x)\}\langle+\infty| - \infty\rangle_J = 0.\end{aligned} \tag{2.4}$$

We here assume the existence of a formally quantized field $\hat{\phi}(x)$, though its detailed properties are not specified yet, and the asymptotic states $|\pm\infty\rangle_J$ in the presence of a source function $J(x)$ localized in space-time. The path integral is then defined as a formal solution of the above functional equation

$$\langle+\infty| - \infty\rangle_J = \int\mathcal{D}\phi\exp\{i\int d^4x\mathcal{L}_J\}. \tag{2.5}$$

We now define the Green's function (correlation function) by

$$\begin{aligned}\langle+\infty|T^\star\hat{\phi}(x)\hat{\phi}(y)| - \infty\rangle &= \frac{\delta}{i\delta J(x)}\frac{\delta}{i\delta J(y)}\langle+\infty| - \infty\rangle_J|_{J=0} \\ &= \frac{1}{i}\frac{1}{\Box - i\epsilon - \lambda\Box^2}\delta(x-y).\end{aligned} \tag{2.6}$$

This Green's function contains all the information about the quantized field.

The BJL prescription states that we can replace the covariant $T^\star$ product by the conventional T product when

$$\lim_{k^0\to\infty}\int d^4xe^{ik(x-y)}\langle+\infty|T^\star\hat{\phi}(x)\hat{\phi}(y)| - \infty\rangle = \lim_{k^0\to\infty}\frac{i}{k^2 + i\epsilon + \lambda(k^2)^2} = 0. \tag{2.7}$$

An elementary account of the BJL prescription is given in the Appendix of Ref.[17], for example. Thus we have

$$\int d^4xe^{ik(x-y)}\langle+\infty|T\hat{\phi}(x)\hat{\phi}(y)| - \infty\rangle = \frac{i}{k^2 + i\epsilon + \lambda(k^2)^2}. \tag{2.8}$$

[1] Our metric convention is $g_{\mu\nu} = (1, -1, -1, -1)$.

By multiplying a suitable powers of the momentum variable k_μ, we can recover the canonical commutation relations. For example,

$$
\begin{aligned}
& k_\mu \int d^4x e^{ik(x-y)} \langle +\infty | T\hat{\phi}(x)\hat{\phi}(y)| -\infty\rangle \\
&= \int d^4x (-i\partial^x_\mu e^{ik(x-y)}) \langle +\infty | T\hat{\phi}(x)\hat{\phi}(y)| -\infty\rangle \\
&= \int d^4x e^{ik(x-y)} \{ \langle +\infty | Ti\partial^x_\mu \hat{\phi}(x)\hat{\phi}(y)| -\infty\rangle \\
&\qquad + i\delta(x^0 - y^0) \langle +\infty | [\hat{\phi}(x), \hat{\phi}(y)] | -\infty\rangle \} \\
&= \frac{ik_\mu}{k^2 + i\epsilon + \lambda(k^2)^2}.
\end{aligned} \tag{2.9}
$$

An analysis of this relation in the limit $k_0 \to \infty$ gives

$$
\begin{aligned}
& \delta(x^0 - y^0)[\hat{\phi}(x), \hat{\phi}(y)] = 0, \\
& \int d^4x e^{ik(x-y)} \{ \langle +\infty | Ti\partial^x_\mu \hat{\phi}(x)\hat{\phi}(y)| -\infty\rangle = \frac{ik_\mu}{k^2 + i\epsilon + \lambda(k^2)^2}.
\end{aligned} \tag{2.10}
$$

Note that the limit $k_0 \to \infty$ of the Fourier transform of a T product such as $\langle +\infty | Ti\partial^x_\mu \hat{\phi}(x)\hat{\phi}(y)| - \infty\rangle$ vanishes *by definition*.

By repeating the procedure with (2.10), we obtain

$$
\begin{aligned}
& \delta(x^0 - y^0)[\partial_0\hat{\phi}(x), \hat{\phi}(y)] = 0, \\
& \int d^4x e^{ik(x-y)} \langle +\infty | T(i)^2\Box\hat{\phi}(x)\hat{\phi}(y)| -\infty\rangle = \frac{ik^2}{k^2 + i\epsilon + \lambda(k^2)^2}
\end{aligned} \tag{2.11}
$$

and

$$
\begin{aligned}
& \delta(x^0 - y^0)[\partial_0^2\hat{\phi}(x), \hat{\phi}(y)] = 0, \\
& \int d^4x e^{ik(x-y)} \langle +\infty | T(i)^3\partial_\mu\Box\hat{\phi}(x)\hat{\phi}(y)| -\infty\rangle = \frac{ik_\mu k^2}{k^2 + i\epsilon + \lambda(k^2)^2}.
\end{aligned} \tag{2.12}
$$

The final step then gives

$$
\begin{aligned}
\delta(x^0 - y^0)[\partial_0^3\hat{\phi}(x), \hat{\phi}(y)] &= \frac{i}{\lambda}, \\
\int d^4x e^{ik(x-y)} \langle +\infty | T(i)^4\Box^2\hat{\phi}(x)\hat{\phi}(y)| -\infty\rangle &= \frac{i(k^2)^2}{k^2 + i\epsilon + \lambda(k^2)^2} - \frac{i}{\lambda} \\
&= -\frac{i}{\lambda}\frac{k^2}{k^2 + i\epsilon + \lambda(k^2)^2}.
\end{aligned} \tag{2.13}
$$

The last relation can be written by using (2.11) as

$$
\int d^4x e^{ik(x-y)} \langle +\infty | T\{[\lambda\Box^2 + \Box]\hat{\phi}(x)\}\hat{\phi}(y)| -\infty\rangle = 0 \tag{2.14}
$$

which is consistent with

$$\langle +\infty | T^\star \{[\lambda \Box^2 + \Box]\hat{\phi}(x)\}\hat{\phi}(y)| -\infty\rangle = i\delta^{(4)}(x-y) \tag{2.15}$$

derived from the path integral, when combined with the definitions of T and $T^\star$ products.

We thus derived the canonical commutation relations for the higher derivative theory

$$\begin{aligned}
&\delta(x^0-y^0)[\hat{\phi}(x),\hat{\phi}(y)] = 0,\\
&\delta(x^0-y^0)[\partial_0\hat{\phi}(x),\hat{\phi}(y)] = 0,\\
&\delta(x^0-y^0)[\partial_0^2\hat{\phi}(x),\hat{\phi}(y)] = 0,\\
&\delta(x^0-y^0)[\partial_0^3\hat{\phi}(x),\hat{\phi}(y)] = \frac{i}{\lambda}\delta^{(4)}(x-y).
\end{aligned} \tag{2.16}$$

We can also confirm from (2.11) by considering the derivative with respect to the variable y^μ and by following the procedure similar to the above

$$\begin{aligned}
&\delta(x^0-y^0)[\partial_0\hat{\phi}(x),\partial_0\hat{\phi}(y)] = 0,\\
&\delta(x^0-y^0)[\partial_0^2\hat{\phi}(x),\partial_0\hat{\phi}(y)] = -\frac{i}{\lambda}\delta^{(4)}(x-y).
\end{aligned} \tag{2.17}$$

The general rule is that the commutator

$$[\hat{\phi}^{(m)}(x),\hat{\phi}^{(l)}(y)]\delta(x^0-y^0) \neq 0 \tag{2.18}$$

where $m+l = n-1$ for a theory with the n-th time derivative. Here $\hat{\phi}^{(l)}(x)$ stands for the l-th time derivative of $\hat{\phi}(x)$

$$\hat{\phi}^{(l)}(x) = \frac{\partial^l}{\partial (x^0)^l}\hat{\phi}(x). \tag{2.19}$$

We thus derive all the canonical commutation relations (2.16) and (2.17) from the path integral defined by the Schwinger's action principle and the $T^\star$ product, and those commutation relations naturally agree with the relations derived by a canonical formulation of the higher derivative theory[16]. A crucial property of the higher derivative theory is that the canonical commutation relations are defined by the "term with the highest derivative" with the parameter λ. The quantization with a naive picture with $\lambda = 0$ even for a small parameter λ does not correctly describe even the qualitative features of the quantized theory. It is well-known that the above higher derivative theory contains a negative norm state, and thus not unitary.

We next comment on a higher derivative theory defined by

$$\mathcal{L}_J = \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \lambda\phi(x)(\Box\phi(x))(\Box\phi(x)) + J(x)\phi(x). \tag{2.20}$$

In this case, one can confirm that the one-loop diagrams (in a naive formulation of perturbation theory) induce a divergence corresponding to the term $\phi(x)(\Box^2\phi(x))$. This suggests that a consistent theory needs to be formulated at least with

$$\mathcal{L}_J = \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) + \lambda_1\phi(x)(\Box^2\phi(x)) - \lambda\phi(x)(\Box\phi(x))(\Box\phi(x)) + J(x)\phi(x) \tag{2.21}$$

with a suitable constant λ_1 from the beginning. We thus arrive at the case we analyzed above in (2.1). Namely, the higher derivative terms in the interaction generally lead to the problem of quantizing higher derivative theory. (This is also the case for a non-renormalizable extension of the supersymmetric Wess-Zumino model where a higher derivative Kähler term is induced[18].) The canonical analysis of such a theory is involved, but the path integral analysis is relatively easier as illustrated above, in addition to giving a simple path integral formula for correlation functions defined by the $T^\star$ product.

3 Quantization of a theory non-local in time

We examine a non-local theory defined by

$$\begin{aligned}\mathcal{L}_J &= -\frac{1}{2}\phi(x)\Box[\phi(x+\xi)+\phi(x-\xi)]+J(x)\phi(x)\\ &= -\frac{1}{2}\phi(x)\Box[e^{i\xi\hat{p}}+e^{-i\xi\hat{p}}]\phi(x)+J(x)\phi(x).\end{aligned} \tag{3.1}$$

This Lagrangian is somewhat analogous to a lattice theory, but we treat this Lagrangian as a non-local theory defined in continuum. A formal integration of the Schwinger's action principle

$$\begin{aligned}&\langle +\infty| -\Box[\hat{\phi}(x+\xi)+\hat{\phi}(x-\xi)]+J(x)|-\infty\rangle_J\\ &=\{-\Box[\frac{\delta}{i\delta J(x+\xi)}+\frac{\delta}{i\delta J(x-\xi)}]+J(x)\}\langle +\infty|-\infty\rangle_J=0\end{aligned} \tag{3.2}$$

gives a path integral

$$\langle +\infty|-\infty\rangle_J=\int\mathcal{D}\phi\exp\{i\int d^4x\mathcal{L}_J\}, \tag{3.3}$$

which in turn leads to the correlation function

$$\langle T^\star\hat{\phi}(x)\hat{\phi}(y)\rangle=\int\frac{d^4k}{(2\pi)^4}\frac{i}{(k^2+i\epsilon)[e^{ik\xi}+e^{-ik\xi}]}e^{-ik(x-y)} \tag{3.4}$$

or

$$\int d^4x e^{ik(x-y)}\langle T^\star\hat{\phi}(x)\hat{\phi}(y)\rangle=\frac{i}{(k^2+i\epsilon)[e^{ik\xi}+e^{-ik\xi}]}. \tag{3.5}$$

For a time-like vector ξ, which may be chosen as $(\xi^0,0,0,0)$, the right-hand side of this expression multiplied by any power of k_0 goes to zero

$$\lim_{k_0\to i\infty}\frac{i(k_0)^n}{(k^2+i\epsilon)[e^{ik\xi}+e^{-ik\xi}]}=0 \tag{3.6}$$

for k_0 along the imaginary axis in the complex k_0 plane[2]. Thus the application of BJL prescription leads to (for any pair of non-negative integers n and m)

$$[\hat{\phi}^{(n)}(x),\hat{\phi}^{(m)}(y)]\delta(x^0-y^0)=0 \tag{3.7}$$

[2]For general cases, we take k_0 along the imaginary axis as is suggested by a smooth Wick rotation to avoid possible singularities; for a theory analyzed in the previous section, this careful choice of the direction of k_0 was not required.

where $\hat{\phi}^{(n)}(x)$ stands for the n-th time derivative of $\hat{\phi}(x)$ as in (2.19). This relation is consistent with the $N \to \infty$ limit of a higher derivative theory obtained by a truncation of the power series exapansion of $e^{\pm i\xi\hat{p}}$ at the N-th power in the starting Lagrangian (3.1). See also the analysis in the previous section. In contrast, for a space-like ξ for which one may choose $\xi = (0, \vec{\xi})$, one recovers the result of the naive canonical quantization of (3.1)

$$\begin{aligned}
&\delta(x^0 - y^0)[\hat{\phi}(x), \hat{\phi}(y)] = 0, \\
&\delta(x^0 - y^0)[\partial_0\hat{\phi}(x), \hat{\phi}(y)] = \frac{-i}{[e^{i\xi\hat{p}} + e^{-i\xi\hat{p}}]}\delta^{(4)}(x - y)
\end{aligned} \tag{3.8}$$

by means of BLJ prescription; in the right-hand side of (3.8), $\hat{p}$ stands for the (spatial) momentum operator acting on the coordinate $\vec{x}$.

We can thus define no sensible canonical structure for the present non-local theory for a time-like ξ. Nevertheless, we can fomally define a quantum theory by the Schwinger's action principle and the path integral. The quantization is defined by a specification of $\langle T^\star \hat{\phi}(x)\hat{\phi}(y)\rangle$.

We next analyze a theory which contains a non-local interaction

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{m^2}{2}\phi(x)\phi(x) \\
&\quad -\frac{g}{2}[\phi(x+\xi)\phi(x)\phi(x-\xi) + \phi(x-\xi)\phi(x)\phi(x+\xi)] + \phi(x)J(x) \\
&= \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{m^2}{2}\phi(x)\phi(x) \\
&\quad -\frac{g}{2}[(e^{\xi^\mu\partial_\mu}\phi(x))\phi(x)(e^{-\xi^\mu\partial_\mu}\phi(x)) + (e^{-\xi^\mu\partial_\mu}\phi(x))\phi(x)(e^{\xi^\mu\partial_\mu}\phi(x))] \\
&\quad +\phi(x)J(x)
\end{aligned} \tag{3.9}$$

where ξ^μ is a constant four-vector. This theory is not Lorentz invariant because of the constant vector ξ^μ. When one chooses ξ^μ to be a time-like vector $\xi^2 = (\xi^0)^2 - (\vec{\xi})^2 > 0$, the quantization of the above theory is analogous to that of space-time noncommutative theory. When one works in the frame

$$\xi^\mu = (\xi^0, 0, 0, 0) \;\; \text{with} \;\; \xi^0 > 0 \tag{3.10}$$

which we adopt in the rest of this section, it is obvious that the unitary time development (in the sense of the Schrödinger equation) for the small time interval $\Delta t < \xi^0$ is not defined. One may examine a naive Hamiltonian

$$\begin{aligned}
\mathcal{H} &= \frac{1}{2}\Pi^2(x) + \frac{1}{2}\vec{\nabla}\phi(x)\vec{\nabla}\phi(x) + \frac{1}{2}m^2\phi^2(x) \\
&\quad +\frac{g}{2}[\phi(x-\xi)\phi(x)\phi(x+\xi) + \phi(x+\xi)\phi(x)\phi(x-\xi)]
\end{aligned} \tag{3.11}$$

where $x^\mu = (0, \vec{x})$ and $\xi^\mu = (\xi^0, \vec{0})$, and $\Pi(x) = \frac{\partial}{\partial x^0}\phi(x)$ is a naive canonical momentum conjugate to $\phi(x)$. This Hamiltonian is formally hermitian, $\mathcal{H}^\dagger = \mathcal{H}$, but $\mathcal{H}$ is not local

in the time coordinate and does not generate time development in the conventional sense for the small time interval $\Delta t < \xi^0$. The equal-time commutation relation, for example,

$$[\int d^3x\mathcal{H}(x), \phi(y)]\delta(x^0 - y^0) \tag{3.12}$$

is not well defined, since $[\phi(x + \xi), \phi(y)]\delta(x^0 - y^0)$ is not well specified in the non-perturbative level.

Nevertheless, one may study the path integral quantization without specifying the precise quantization condition of field variables. This aspect is analogous to the Yang-Feldman formulation. One may thus define a path integral by means of Schwinger's action principle and a suitable ansatz of asymptotic conditions as in (3.2)

$$\langle +\infty | - \infty\rangle_J = \int \mathcal{D}\phi \exp[i \int d^4x\mathcal{L}_J]. \tag{3.13}$$

One may then define a formal expansion in powers of the coupling constant g. It is interesting to examine what one learns as to the canonical quantization and unitarity relations defined by the Feynman diagrams.

We study one-loop diagrams in a formal perturbative expansion in powers of the coupling constant g by starting with a tentative ansatz of quantization

$$\langle T^\star\hat{\phi}(x)\hat{\phi}(y)\rangle = \frac{-i}{\Box + m^2 - i\epsilon}\delta(x - y) = \int \frac{d^4k}{(2\pi)^4}e^{-ik(x-y)}\frac{i}{k^2 - m^2 + i\epsilon} \tag{3.14}$$

which is equivalent to a canonical quantization of free theory. One-loop self-energy diagrams contain the contributions

$$\begin{aligned}
&\frac{(-ig)^2}{2}\int d^4xd^4y\phi(x)\langle T^\star\phi(x + \xi)\phi(x - \xi)\phi(y + \xi)\phi(y - \xi)\rangle\phi(y)\\
&= \frac{(-ig)^2}{2}\int d^4xd^4y\phi(x)\phi(y)[\langle T^\star\phi(x + \xi)\phi(y + \xi)\rangle\langle T^\star\phi(x - \xi)\phi(y - \xi)\rangle\\
&\quad+\langle T^\star\phi(x + \xi)\phi(y - \xi)\rangle\langle T^\star\phi(x - \xi)\phi(y + \xi)\rangle]
\end{aligned} \tag{3.15}$$

The first term in (3.15) gives rise to a logarithmically divergent local contribution, which is absorbed into the mass renormalization, and the second term gives a finite non-local (approximately separated by $\sim 2\xi$) term. We also have contributions

$$\begin{aligned}
&\frac{(-ig)^2}{2}\int d^4xd^4y\phi(x)\langle T^\star\phi(x + \xi)\phi(x - \xi)\phi(y)\phi(y - \xi)\rangle\phi(y + \xi)\\
&= \frac{(-ig)^2}{2}\int d^4xd^4y\phi(x)\phi(y)[\langle T^\star\phi(x + \xi)\phi(y - \xi)\rangle\langle T^\star\phi(x - \xi)\phi(y - 2\xi)\rangle\\
&\quad+\langle T^\star\phi(x + \xi)\phi(y - 2\xi)\rangle\langle T^\star\phi(x - \xi)\phi(y - \xi)\rangle]
\end{aligned} \tag{3.16}$$

which contains the finite non-local terms separated up to the order of $\sim 3\xi$.

The first term in (3.16), for example, gives rise to

$$g^2 i\Sigma(k, \xi)$$

$$
\begin{aligned}
&= \frac{-g^2}{2}\int \frac{d^4k}{(2\pi)^4}[e^{2ik\xi}e^{i\xi(p-k)} + e^{-2ik\xi}e^{-i\xi(p-k)}]\frac{i}{k^2-m^2+i\epsilon}\frac{i}{(p-k)^2-m^2+i\epsilon} \\
&= \frac{-g^2}{2}\int \frac{d^4k}{(2\pi)^4}[e^{i\xi(p+k)} + e^{-i\xi(p+k)}]\frac{i}{k^2-m^2+i\epsilon}\frac{i}{(p-k)^2-m^2+i\epsilon} \\
&= \frac{-g^2}{2}\int \frac{d^4k}{(2\pi)^4}[e^{i\xi(p+k)} + e^{-i\xi(p+k)}]\int_0^\infty dz_1 dz_2 e^{iz_1[k^2-m^2+i\epsilon]+iz_2[(p-k)^2-m^2+i\epsilon]}. \quad (3.17)
\end{aligned}
$$

Note that the Feynman's $m^2 - i\epsilon$ prescrition provides a convergent factor at $z_{1,2} = \infty$. One can further evaluate this by setting $z_1 = \alpha x$ and $z_2 = \alpha(1-x)$ as

$$
\begin{aligned}
& g^2 i\Sigma(k,\xi) \\
&= \frac{-g^2}{2}\int \frac{d^4k}{(2\pi)^4}\int_0^\infty \alpha d\alpha \int_0^1 dx e^{i\alpha[k^2+x(1-x)p^2-m^2+i\epsilon)]} \\
&\quad \times [e^{i(2-x)\xi p+i\xi k} + e^{-i(2-x)\xi p-i\xi k}] \\
&= \frac{ig^2}{2(4\pi)^2}\int_0^1 dx[e^{i(2-x)\xi p} + e^{-i(2-x)\xi p}]\int_0^\infty \frac{d\alpha}{\alpha}e^{i\alpha[x(1-x)p^2-m^2+i\epsilon)]-i\frac{\xi^2}{4\alpha}}. \quad (3.18)
\end{aligned}
$$

We analyze the part of the above amplitude

$$
\int_0^\infty \frac{d\alpha}{\alpha}e^{i\alpha[x(1-x)p^2-m^2+i\epsilon)]-i\frac{\xi^2}{4\alpha}}. \quad (3.19)
$$

Following the conventional approach, we define the integral for a Euclidean momentum p_μ, for which $p^2 < 0$. In this case, one can deform the integration contour along the negative real axis as

$$
\begin{aligned}
&\int_0^{\infty e^{-i\pi}} \frac{d\alpha}{\alpha}e^{i\alpha[x(1-x)p^2-m^2+i\epsilon)]-i\frac{\xi^2}{4\alpha}} \\
&= \int_0^{\infty e^{-i\pi}} \frac{d\alpha}{\alpha}e^{-i\frac{\sqrt{(-\xi^2)[-x(1-x)p^2+m^2-i\epsilon)}}{2}[\alpha-\frac{1}{\alpha}]} \\
&= -i\pi H_0^{(2)}(-i\sqrt{(-\xi^2)[-x(1-x)p^2+m^2-i\epsilon)}) \quad (3.20)
\end{aligned}
$$

for $\xi^2 < 0$. Here $H_0^{(2)}(z)$ stands for the Hankel function which has an asymptotic expansion for $|z| \to \infty$

$$
H_0^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}}e^{-i(z-\frac{\pi}{4})} \quad (3.21)
$$

for $-2\pi < \arg z < \pi$.

We thus find that for $p_0 \to i\infty$

$$
\int_0^\infty \frac{d\alpha}{\alpha}e^{i\alpha[x(1-x)p^2-m^2+i\epsilon)]-i\frac{\xi^2}{4\alpha}} \sim -\pi\sqrt{\frac{2}{\pi z}}e^{-z} \quad (3.22)
$$

with

$$
z = \sqrt{(-\xi^2)[-x(1-x)p^2+m^2-i\epsilon)} \quad (3.23)
$$

for a space-like ξ, $\xi^2 < 0$. On the other hand, we have a damping oscillatory behavior for $p_0 \to i\infty$,

$$\int_0^\infty \frac{d\alpha}{\alpha} e^{i\alpha[x(1-x)p^2 - m^2 + i\epsilon)] - i\frac{\xi^2}{4\alpha}} \sim -i\pi\sqrt{\frac{2}{\pi z}} e^{(-iz + i\frac{\pi}{4})} \tag{3.24}$$

with

$$z = \sqrt{(\xi^2)[-x(1-x)p^2 + m^2 - i\epsilon)} \tag{3.25}$$

for a time-like ξ, $\xi^2 > 0$, which is defined by an analytic continuation.

When one writes the (complete) connected two-point correlation function with one-loop corrections as

$$\langle T^\star \hat{\phi}(x)\hat{\phi}(y)\rangle_{ren} = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{i}{p^2 + g^2\Sigma(p,\xi) - m_r^2 + i\epsilon}, \tag{3.26}$$

the two-point function generally contains the non-local term in $g^2\Sigma(p,\xi)$. When one applies the BJL prescription to the two-point correlation function in a conventional local renormalizable theory, the higher order corrections do not modify the canonical structure since we apply the BJL prescription to the two-point function with the ultraviolet cut-off of loop momenta kept fixed. In the present context this corresponds to the replacement

$$\begin{aligned} &\int_0^\infty \frac{d\alpha}{\alpha} e^{i\alpha[x(1-x)p^2 - m^2 + i\epsilon)] - i\frac{\xi^2}{4\alpha}} \\ &\to \int_0^\infty \frac{d\alpha}{\alpha} e^{i\alpha[x(1-x)p^2 - m^2 + i\epsilon)] - i\frac{\xi^2}{4\alpha}} - \int_0^\infty \frac{d\alpha}{\alpha} e^{i\alpha[x(1-x)p^2 - M^2 + i\epsilon)] - i\frac{\xi^2}{4\alpha}} \end{aligned} \tag{3.27}$$

where M stands for the Pauli-Villars-type cut-off mass.

In the present case also, $g^2\Sigma(p,\xi)$ assumes real values and $g^2\Sigma(p,\xi) \to 0$ for $p_0 \to i\infty$ for the space-like ξ as in (3.22), for which we may take $\xi = (0, \vec{\xi})$. The canonical structure is not modified by the one-loop effects of the interaction non-local in the spatial distance.

In contrast, for the time-like ξ for which we may take $\xi = (\xi^0, \vec{0})$, $g^2\Sigma(p,\xi)$ diverges exponentially for $p_0 \to i\infty$. This arises from the behavior of the factor

$$[e^{i(2-x)\xi p} + e^{-i(2-x)\xi p}] \tag{3.28}$$

in (3.18) for $p_0 \to i\infty$ and $\xi = (\xi^0, \vec{0})$, which dominates the damping oscillatory behavior (3.24)[3]. The canonical structure specified by the BJL analysis is thus completely modified by the one-loop effects of the interaction non-local in time. After one-loop corrections, we essentially have the same result (3.6) as for the non-local theory (3.1). The naive ansatz of the two-point correlation function at the starting point of perturbation theory (3.14) is not justified. We thus conclude that the present model for a time-like ξ does not accommodate a consistent canonical structure of quantized theory. In contrast, the naive ansatz (3.14) is not modified by the one-loop quantum corrections for a space-like ξ.

[3]The non-vanishing imaginary part of $g^2\Sigma(p,\xi)$ in (3.24) for the Euclidean momentum given by $p_0 \to i\infty$ is associated with the violation of unitarity in the present theory non-local in time [8, 9, 10].

Nevertheless, it is instructive to examine the formal perturbative unitarity of an S-matrix defined for the theory non-local in time. One may first observe that

$$S(t_+, t_-) = e^{i\hat{H}_0 t_+} e^{-i\hat{H}(t_+ - t_-)} e^{-i\hat{H}_0 t_-} \tag{3.29}$$

for $\mathcal{H}$ in (3.11) with

$$H_0 = \int d^3x [\frac{1}{2}\Pi^2(0,\vec{x}) + \frac{1}{2}\vec{\nabla}\phi(0,\vec{x})\vec{\nabla}\phi(0,\vec{x}) + \frac{1}{2}m^2\phi^2(0,\vec{x})] \tag{3.30}$$

is unitary

$$S(t_+, t_-)^\dagger S(t_+, t_-) = S(t_+, t_-) S(t_+, t_-)^\dagger = 1 \tag{3.31}$$

The formal power series expansion in the coupling constant[4]

$$S(t_+, t_-) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_-}^{t_+} dt_1 .... \int_{t_-}^{t_+} dt_n T_\star(\hat{H}_I(t_1)....\hat{H}_I(t_n)) \tag{3.32}$$

with a hermitian

$$\hat{H}_I(t) = e^{i\hat{H}_0 t} \int d^3x \frac{g}{2} [\phi(-\xi,\vec{x})\phi(0,\vec{x})\phi(\xi,\vec{x}) + \phi(\xi,\vec{x})\phi(0,\vec{x})\phi(-\xi,\vec{x})] e^{-i\hat{H}_0 t} \tag{3.33}$$

thus defines a unitary operator

$$\hat{S} = \lim_{t_- \to -\infty, t_+ \to +\infty} S(t_+, t_-). \tag{3.34}$$

This definition of a unitary operator corresponds to the definition of a unitary S-matrix for space-time noncommutative theory proposed in [11, 12].

It is important to recognize that the time-ordering in the present context is defined with repect to the time variable appearing in $\hat{H}_I(t)$; if one performs a time-ordering with repect to the time variable appearing in each field variable $\phi(x)$, one generally obtains different results due to the non-local structure of the interaction term in time. Since the operator $\hat{S}$ defined above is manifestly unitary, the non-unitary result in the conventional Feynman rules, which are based on the time-ordering of each operator $\phi(x)$, arises from this difference of time ordering. In any case, it should be possible to understand the origin of unitary or non-unitary S-matrix in the coordinate representation without recourse to the momentum representation of Feynman diagrams.

When one defines

$$\begin{aligned} A_1 &= \int_{-\infty}^{\infty} dt \hat{H}_I(t), \\ A_2 &= \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt_1 dt_2 T_\star \hat{H}_I(t_1) \hat{H}_I(t_2) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt_1 dt_2 \theta(t_1 - t_2) \hat{H}_I(t_1) \hat{H}_I(t_2) \end{aligned} \tag{3.35}$$

[4]We use the notation $T_\star$ for the time ordering in non-local theory, whereas T or $T^\star$ is used for the conventional time ordering with respect to the time variable of *each* field variable $\phi(x)$.

the unitarity relation of the above S-matrix in the second order of the coupling constant requires(see, for example, [12])

$$A_2 + A_2^\dagger = A_1^\dagger A_1 = A_1^2. \tag{3.36}$$

To be explicit

$$\begin{aligned} A_2 + A_2^\dagger &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt_1 dt_2 \theta(t_1 - t_2)\{\hat{H}_I(t_1)\hat{H}_I(t_2) + \hat{H}_I(t_2)\hat{H}_I(t_1)\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt_1 dt_2 \{\theta(t_1 - t_2)\hat{H}_I(t_1)\hat{H}_I(t_2) + \theta(t_2 - t_1)\hat{H}_I(t_1)\hat{H}_I(t_2)\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt_1 dt_2 \hat{H}_I(t_1)\hat{H}_I(t_2) \\ &= \int_{-\infty}^{\infty} dt_1 \hat{H}_I(t_1) \int_{-\infty}^{\infty} dt_2 \hat{H}_I(t_2) \\ &= A_1^2 \end{aligned} \tag{3.37}$$

by noting $\theta(t_1 - t_2) + \theta(t_2 - t_1) = 1$, as required by the unitarity relation.

In contrast, if one uses the conventional time ordering one has

$$\begin{aligned} A_2 &= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt_1 dt_2 T^\star \hat{H}_I(t_1)\hat{H}_I(t_2) \\ &\neq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt_1 dt_2 \theta(t_1 - t_2)\hat{H}_I(t_1)\hat{H}_I(t_2) \end{aligned} \tag{3.38}$$

since the time ordering by $T^\star$ is defined with respect to the time variable of each field $\phi(t, \vec{x})$, and thus the unitarity of the conventional operator

$$\hat{S} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} dt_1 .... \int_{-\infty}^{+\infty} dt_n T^\star(\hat{H}_I(t_1)....\hat{H}_I(t_n)) \tag{3.39}$$

is not satisfied for the non-local $\hat{H}_I(t)$ in general. Note that the perturbative expansion with the $T^\star$ product is directly defined by the path integral without recourse to the expression such as (3.29).

On the other hand, the positive energy condition, which is ensured by the Feynman propagator, is not obvious for the propagator defined by $T_\star$. To be specific, we have the following correlation function in the Wick-type reduction of the S-matrix

$$\begin{aligned} &\langle 0|T\phi(x - \xi)\phi(y + \xi)|0\rangle \\ &= \int \frac{d^4k}{(2\pi)^4} \exp[-ik((x - \xi) - (y + \xi))] \frac{i}{k_\mu k^\mu - m^2 + i\epsilon} \\ &= \theta((x - \xi)^0 - (y + \xi)^0) \int \frac{d^3k}{(2\pi)^3 2\omega} \exp[-i\omega((x - \xi)^0 - (y + \xi)^0) + i\vec{k}(\vec{x} - \vec{y})] \\ &+\theta((y + \xi)^0 - (x - \xi)^0) \int \frac{d^3k}{(2\pi)^3 2\omega} \exp[-i\omega((y + \xi)^0 - (x - \xi)^0) + i\vec{k}(\vec{y} - \vec{x})] \end{aligned} \tag{3.40}$$

with $\omega = \sqrt{\vec{k}^2 + m^2}$ for the conventional Feynman prescription with $m^2 - i\epsilon$, which ensures that the positive frequency components propagate in the forward time direction and the negative frequency components propagate in the backward time direction and thus the positive energy flows always in the forward time direction. The Wick rotation to Euclidean theory in the momentum space is also smooth in this prescription. The path integral with respect to the field variable $\phi(x)$ gives this time ordering or the $T^\star$ product.

In comparison, the non-local prescription (3.32) gives the following correlation function for the quantized free field in the Wick-type reduction

$$\begin{aligned}&\langle 0|T_\star \phi(x-\xi)\phi(y+\xi)|0\rangle \\ &= \theta(x^0 - y^0)\int \frac{d^3k}{(2\pi)^3 2\omega}\exp[-i\omega((x-\xi)^0 - (y+\xi)^0)) + i\vec{k}(\vec{x}-\vec{y})] \\ &+\theta(y^0 - x^0)\int \frac{d^3k}{(2\pi)^3 2\omega}\exp[-i\omega((y+\xi)^0 - (x-\xi)^0) + i\vec{k}(\vec{y}-\vec{x})]\end{aligned} \tag{3.41}$$

where the time-ordering step function $\theta(x^0 - y^0)$, for example, and the signature of the time variable $(x-\xi)^0 - (y+\xi)^0$ appearing in the exponential are not correlated, and it allows the negative energy to propagate in the forward time direction also. This result is not reproduced by the Feynman's $m^2 - i\epsilon$ prescription. When one considers an arbitrary fixed time-slice in 4-dimesional space-time, the condition that all the particles crossing the time-slice carry the positive energy in the forward time direction, which is regarded as the positive energy condition in the path integral formulation[19] (or in perturbation theory in general), is not satisfied [5]. This positive energy condition is crucial in the analysis of spin-statistics theorem[19, 20], for example. See also [21] for an analysis of spin-statistics theorem in noncommutative theory.

We thus summarize the analysis of this section as follows: The naive canonical quantization in a perturbative sense is not justified in the present theory non-local in time when one incorporates the higher order corrections. The unitarity of the (formal) perturbative S-matrix is ensured if one adopts the $T_\star$ product, but the positive energy condition is not satisfied by this prescription. Also the Wick rotation is not obvious in this modified $T_\star$ product. On the other hand, the unitarity of the S matrix is spoiled if one adopts the conventional T or $T^\star$ product which is defined by the path integral, though the positive energy condition and a smooth Wick rotation are ensured.

[5] The present theory is a generalization of the globally unstable ϕ^3 potential. The positive energy condition we are discussing is independent of this global structure of the potential, and our analysis is valid for the ϕ^4-type potential also.

4 Quantization of space-time noncommutative theory

We study the simplest noncommutative theory defined by

$$\begin{aligned}\mathcal{L}_J &= \frac{1}{2}\partial_\mu\phi(x)\star\partial^\mu\phi(x) - \frac{m^2}{2}\phi(x)\star\phi(x)\\ &\quad -\frac{g}{3!}\phi(x)\star\phi(x)\star\phi(x) + \phi(x)\star J(x)\end{aligned} \tag{4.1}$$

where the $\star$ product is defined by the so-called Moyal product

$$\phi(x)\star\phi(x) = e^{\frac{i}{2}\xi\partial^x_\mu\theta^{\mu\nu}\partial^y_\nu}\phi(x)\phi(y)|_{y=x} = e^{\frac{i}{2}\xi\partial_x\wedge\partial_y}\phi(x)\phi(y)|_{y=x}. \tag{4.2}$$

The real positive parameter ξ stands for the deformation parameter, and the antisymmetric parameter $\theta^{\mu\nu} = -\theta^{\nu\mu}$ corresponds to $i\xi\theta^{\mu\nu} = [\hat{x}^\mu, \hat{x}^\nu]$; since this theory is not Lorentz covariant we consider the case $\theta^{0i} = -\theta^{i0} \neq 0$ for a suitable i but all others $\theta^{\mu\nu} = 0$ in the following.

The formal quantum equation of motion is given by

$$-\Box\hat{\phi}(x) - m^2\hat{\phi}(x) - \frac{g}{2!}\hat{\phi}(x)\star\hat{\phi}(x) + J(x) = 0. \tag{4.3}$$

The Yang-Feldman formulation solves this operator equation (with $J = 0$) by imposing suitable boundary conditions at $t = \pm\infty$ and by using the corresponding two-point (free) Green's functions. It may be noted that the validity of these boundary conditions is not obvious in the present noncommutative theory with $\theta^{0i} \neq 0$.

The Schwinger's action principle starts with the relation

$$\begin{aligned}&\langle +\infty| - \Box\hat{\phi}(x) - m^2\hat{\phi}(x) - \frac{g}{2!}\hat{\phi}(x)\star\hat{\phi}(x) + J(x)| - \infty\rangle_J\\ &= [-\Box\frac{\delta}{i\delta J(x)} - m^2\frac{\delta}{i\delta J(x)} - \frac{g}{2!}\frac{\delta}{i\delta J(x)}\star\frac{\delta}{i\delta J(x)} + J(x)]\\ &\quad\times\langle +\infty| - \infty\rangle_J = 0.\end{aligned} \tag{4.4}$$

This relation assumes the existence of the asymptotic states $|\pm\infty\rangle_J$ at $t = \pm\infty$ in the presence of the source function $J(x)$ which has a support in the finite space-time domain. This Schwinger's action principle thus depends on essentially the same set of assumptions as those of the Yang-Feldman formulation.

The path integral is then defined as a formal integral of the Schwinger's action principle (4.4)

$$\langle +\infty| - \infty\rangle_J = \int\mathcal{D}\phi\exp[i\int d^4x\mathcal{L}_J] \tag{4.5}$$

with a "translational invariant" path integral measure

$$\mathcal{D}(\phi+\epsilon) = \mathcal{D}\phi \tag{4.6}$$

where $\epsilon(x)$ is an arbitrary infinitesimal function independent of $\phi(x)$. The condition (4.6) ensures that the Feynman path integral formula satisfies the Schwinger's action principle.

It has been argued [22] that the present theory is renormalizable in the formal perturbative expansion in powers of the coupling constant g starting with

$$\langle T^{\star}\hat{\phi}(x)\hat{\phi}(y)\rangle = \frac{-i}{\Box + m^2 - i\epsilon}\delta(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{i}{k^2 - m^2 + i\epsilon} \tag{4.7}$$

which is equivalent to a canonical quantization of free theory. The one-loop self-energy is given by

$$\begin{aligned} &g^2 i\Sigma(p,\xi) \\ &= \frac{-g^2}{2}\int \frac{d^4k}{(2\pi)^4}\cos^2(\frac{\xi}{2}p\wedge k)\frac{i}{((p-k)^2 - m^2 + i\epsilon)}\frac{i}{(k^2 - m^2 + i\epsilon)} \\ &= \frac{g^2}{4}\int \frac{d^4k}{(2\pi)^4}\frac{1+\cos(\xi p\wedge k)}{((p-k)^2 - m^2 + i\epsilon)(k^2 - m^2 + i\epsilon)} \end{aligned} \tag{4.8}$$

Since the term without the factor $\cos(\xi p \wedge k)$ is identical to the conventional theory, we concentrate on the term with $\cos(\xi p \wedge k)$

$$\begin{aligned} &\frac{-g^2}{8}\int \frac{d^4k}{(2\pi)^4}[e^{i\xi p\wedge k} + e^{-i\xi p\wedge k}]\int_0^\infty \alpha d\alpha \int_0^1 dx e^{i\alpha[k^2 + x(1-x)p^2 - m^2 + i\epsilon]} \\ &= \frac{ig^2}{4(4\pi)^2}\int_0^\infty \frac{d\alpha}{\alpha}\int_0^1 dx e^{i\alpha[x(1-x)p^2 - m^2 + i\epsilon] - i\frac{\xi^2\tilde{p}^2}{4\alpha}} \end{aligned} \tag{4.9}$$

where

$$\tilde{p}^\mu = \theta^{\mu\nu}p_\nu. \tag{4.10}$$

See also (3.18). For space-time noncommutative theory with $\theta^{01} \neq 0$, for example,

$$\tilde{p}^2 \sim p_1^2 - p_0^2 \tag{4.11}$$

and for space-space noncommutative theory with $\theta^{23} \neq 0$, for example,

$$\tilde{p}^2 \sim -p_2^2 - p_3^2. \tag{4.12}$$

We thus obtain by using the result in (3.20)

$$\begin{aligned} &g^2 i\Sigma(p,\xi)_{non-planar} \\ &= \frac{\pi g^2}{4(4\pi)^2}H_0^{(2)}(-i\sqrt{(-\xi^2\tilde{p}^2)[-x(1-x)p^2 + m^2 - i\epsilon]}) \end{aligned} \tag{4.13}$$

for $\xi^2\tilde{p}^2 < 0$, namely, for space-space noncommutative theory. For space-time noncommutative theory, for which $\xi^2\tilde{p}^2$ can be positive as well as negative, one defines the amplitude by an analytic continuation.

As for the consistency of the naive quantization (4.7), it is important to analyze the self-energy correction in

$$\langle T^{\star}\hat{\phi}(x)\hat{\phi}(y)\rangle_{ren} = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{i}{p^2 + g^2\Sigma(p,\xi) - m_r^2}. \tag{4.14}$$

If $\Sigma(p,\xi)$ contains a non-local exponential factor when one incorporates higher order quantum corrections, the naive quantization is not justified in the framework of the BJL prescription as we explained for the simple theory non-local in time in the previous section. In the BJL analysis, we need to cut-off the loop momenta as in (3.27).

By using the asymptotic expansion in (3.21), we have $\Sigma(p,\xi)$ which assumes real values and exponentially decreases

$$\Sigma(p,\xi) \sim \int_0^1 dx \sqrt{\frac{2}{\pi z}} e^{-z} \tag{4.15}$$

with

$$z = \sqrt{(-\xi^2\tilde{p}^2)[-x(1-x)p^2 + m^2 - i\epsilon)} \tag{4.16}$$

for $p_0 \to i\infty$ and $\xi^2\tilde{p}^2 < 0$, namely, for space-space noncommutative theory. On the other hand, we have a damping oscillatory behavior

$$\Sigma(p,\xi) \sim i\int_0^1 dx \sqrt{\frac{2}{\pi z}} e^{(-iz+i\frac{\pi}{4})} \tag{4.17}$$

with

$$z = \sqrt{(\xi^2\tilde{p}^2)[-x(1-x)p^2 + m^2 - i\epsilon)} \tag{4.18}$$

for $p_0 \to i\infty$ and $\xi^2\tilde{p}^2 > 0$, namely, for space-time noncommutative theory.

We thus conclude that the naive quantization (4.7), either in space-space or in space-time noncommutative theory, is not modified by the one-loop corrections in the framework of BJL prescription. This is in sharp contrast to the simple theory non-local in time analyzed in the previous section. This difference arises from the fact that

$$p_\mu \theta^{\mu\nu} p_\nu = 0 \tag{4.19}$$

and thus the two-point function, which depends on the single momentum p_μ, does not contain an extra exponential factor in the present space-time noncommutative theory. We thus expect that our result based on the one-loop diagram is valid for higher loop diagrams with a suitable cut-off of loop-momenta. Although our analysis does not justify the naive quantization to the non-perturbative accuray, it provides a basis of the formal perturbative expansion in the present model[23].

As for the perturbative unitarity, we observe that

$$S(t_+,t_-) = e^{i\hat{H}_0 t_+} e^{-i\hat{H}(t_+-t_-)} e^{-i\hat{H}_0 t_-} \tag{4.20}$$

is unitary in the present case also

$$S(t_+,t_-)^\dagger S(t_+,t_-) = S(t_+,t_-)S(t_+,t_-)^\dagger = 1 \tag{4.21}$$

where the total Hamiltonian $\hat{H} = \int d^3x\mathcal{H}$ is defined by

$$\begin{aligned}
\mathcal{H} &= \frac{1}{2}\Pi^2(0,\vec{x}) + \frac{1}{2}\vec{\nabla}\phi(0,\vec{x})\vec{\nabla}\phi(0,\vec{x}) + \frac{1}{2}m^2\phi^2(0,\vec{x}) \\
&+ \frac{g}{2\cdot 3!}[\phi(0,\vec{x})\star\phi(0,\vec{x})\star\phi(0,\vec{x}) + h.c.]
\end{aligned} \tag{4.22}$$

with the naive canonical momentum $\Pi(x) = \frac{\partial}{\partial x^0}\phi(x)$ conjugate to the variable $\phi(x)$. See [24] for a different approach to the Hamiltonian formulation of space-time noncommutative theory. The operator defined by a formal perturbative expansion of (4.20)

$$\begin{aligned}
\hat{S} &= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{+\infty}dt_1....\int_{-\infty}^{+\infty}dt_n T_\star(\hat{H}_I(t_1)....\hat{H}_I(t_n)) \\
&= \sum_{n=0}^{\infty}(-i)^n\int_{-\infty}^{+\infty}dt_1....\int_{-\infty}^{+\infty}dt_n\theta(t_1-t_2)...\theta(t_{n-1}-t_n)(\hat{H}_I(t_1)....\hat{H}_I(t_n))
\end{aligned} \tag{4.23}$$

with

$$\begin{aligned}
\hat{H}_I(t) &\equiv e^{i\hat{H}_0 t}\int d^3x\frac{g}{2\cdot 3!}[\phi(0,\vec{x})\star\phi(0,\vec{x})\star\phi(0,\vec{x}) + h.c.]e^{-i\hat{H}_0 t} \\
&= \int d^3x\frac{g}{2\cdot 3!}[\phi(t,\vec{x})\star\phi(t,\vec{x})\star\phi(t,\vec{x}) + h.c.]
\end{aligned} \tag{4.24}$$

and

$$H_0 = \int d^3x[\frac{1}{2}\Pi^2(0,\vec{x}) + \frac{1}{2}\vec{\nabla}\phi(0,\vec{x})\vec{\nabla}\phi(0,\vec{x}) + \frac{1}{2}m^2\phi^2(0,\vec{x})] \tag{4.25}$$

defines a unitary S-matrix

$$\hat{S}\hat{S}^\dagger = \hat{S}^\dagger\hat{S} = 1. \tag{4.26}$$

Note that the time-ordering in (4.23) is defined with repsect to the time t of $\hat{H}_I(t)$. Because of the Moyal product, the interaction Hamiltonian $\hat{H}_I(t)$ is not local in the time variable. We thus encounter the same complications as in the non-local theory we analyzed in the previous section. The unitary S-matrix thus generally spoils the perturbative positive energy condition.

On the other hand, the conventional S-matrix, which corresponds to the one given by the path integral,

$$\hat{S} = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{+\infty}dt_1....\int_{-\infty}^{+\infty}dt_n T^\star(\hat{H}_I(t_1)....\hat{H}_I(t_n)) \tag{4.27}$$

is based on the time ordering of the time variable appearing in each field variable $\phi(t,\vec{x})$ and, for example, the second order term given by the path integral has the property

$$\begin{aligned}
&\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dt_1dt_2T^\star\hat{H}_I(t_1)\hat{H}_I(t_2) \\
\neq &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dt_1dt_2\theta(t_1-t_2)\hat{H}_I(t_1)\hat{H}_I(t_2)
\end{aligned} \tag{4.28}$$

for the space-time noncommutative theory and thus it is not unitary, though the positive energy condition in the sense that the positive energy always flows in the positive time direction is satisfied. We emphasize that (4.27) is defined directly by the path integral without recourse to the expression such as (4.20).

To be more explicit, we have

$$
\begin{aligned}
&\frac{1}{2}[\phi(x)\star\phi(x)\star\phi(x)+h.c.] \\
&=\cos(\frac{\xi}{2}(\partial_{x_1}\wedge(\partial_{x_2}+\partial_{x_3})+\partial_{x_2}\wedge\partial_{x_3}))\phi(x_1)\phi(x_2)\phi(x_3)|_{x_1=x_2=x_3=x} \\
&=\sum_{p_1,p_2,p_3}\cos(\frac{\xi}{2}(p_1\wedge(p_2+p_3)+p_2\wedge p_3))e^{ip_1x}e^{ip_2x}e^{ip_3x}\phi(p_1)\phi(p_2)\phi(p_3) \\
&=\sum_{p_1,p_2,p_3}\frac{1}{2}[e^{ip_1(x+\frac{\xi}{2}\wedge(p_2+p_3))}e^{ip_2(x+\frac{\xi}{2}\wedge p_3)}e^{ip_3x}+e^{ip_1(x-\frac{\xi}{2}\wedge(p_2+p_3))}e^{ip_2(x-\frac{\xi}{2}\wedge p_3)}e^{ip_3x}] \\
&\times\phi(p_1)\phi(p_2)\phi(p_3)
\end{aligned}
\tag{4.29}
$$

Although the way of writing the last line of the above equation is not unique, it shows that the non-local parameter in the present context is momentum dependent and non-locality becomes more significant for the larger momenta of the neighboring fields.

By using this interaction vertex (4.29), the evaluation of the one-loop self-energy diagram on the basis of the conventional T or $T^\star$ product starts with

$$
\begin{aligned}
&-\frac{1}{2}\frac{g^2}{(3!)^2}\int_{-\infty}^{\infty}d^4xd^4yT^\star\frac{1}{2}[\phi(x)\star\phi(x)\star\phi(x)+h.c.] \\
&\qquad\times\frac{1}{2}[\phi(y)\star\phi(y)\star\phi(y)+h.c.]
\end{aligned}
\tag{4.30}
$$

and this gives rise to

$$
\frac{g^2}{2}\int\frac{d^4l}{(2\pi)^4}\frac{\cos^2(\frac{\xi}{2}p\wedge l)}{((p-l)^2-m^2+i\epsilon)(l^2-m^2+i\epsilon)}
\tag{4.31}
$$

after integrating over the coordinates of the two vertex points, as we have discussed in (4.8). The Feynman's $m^2-i\epsilon$ prescription ensures the conventional time ordering of each field variable $\phi(x)$ by taking into account the momentum dependent non-local effects; the positive energy always flows in the forward time direction by incorporating the momentum dependent non-local effects. It is known that the present expression of the one-loop two point function (4.31) does not satisfy the unitarity relation[8, 9, 10], as is witnessed by the non-vanishing imaginary part of $\Sigma(p,\xi)$ in (4.17) for the Euclidean momentum given by $p_0\to i\infty$. The conventional Wick rotation to Euclidean theory is well-defined because of the $m^2-i\epsilon$ prescription, if one defines a suitable rotation of the wedge product $p\wedge l$.

On the other hand, the non-local prescription starts with

$$
\begin{aligned}
&-\frac{g^2}{(3!)^2}\int_{-\infty}^{\infty}d^4xd^4y\theta(x^0-y^0)\frac{1}{2}[\phi(x)\star\phi(x)\star\phi(x)+h.c.] \\
&\qquad\times\frac{1}{2}[\phi(y)\star\phi(y)\star\phi(y)+h.c.]
\end{aligned}
\tag{4.32}
$$

and the representation

$$\theta(x^0-y^0) = -\int_{-\infty}^{\infty}\frac{d\omega}{2\pi i}\frac{e^{-i\omega(x^0-y^0)}}{\omega+i\epsilon}. \tag{4.33}$$

We thus obtain after extracting the overall 4-momentum conserving δ-function

$$\frac{g^2}{2}\int\frac{d^4l}{(2\pi)^4}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi i(\omega+i\epsilon)}\cos^2(\frac{\xi}{2}p\wedge l)\tilde{\Delta}_+(l)\tilde{\Delta}_+(p-l-\omega) \tag{4.34}$$

where

$$\Delta_+(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle = \int\frac{d^4k}{(2\pi)^4}e^{-ik(x-y)}\tilde{\Delta}_+(k) \tag{4.35}$$

and $\tilde{\Delta}_+(k) = 2\pi\delta(k^2-m^2)\theta(k^0)$; the variable ω in $p-l-\omega$ stands for $(\omega,0,0,0)$. It is known that this expression (4.34) satisfies the unitarity relation[11, 12] though the energy-momentum conservation, which is a result of the translational invariance of the starting action, is not manifest in the present notation. The time ordering in the present case (4.32) is specified by $\theta(x^0-y^0)$ in front of the Moyal products, and thus the time ordering of each field variable $\phi(x)$ induced by the space-time noncommutative product is ignored. Also, a smooth Wick rotation to Euclidean theory is not obvious.

5 Discussion

We illustrated that the path integral on the basis of Schwinger's action principle has a wide range of applications. The time ordering of field operators is rigidly specified to be the conventional one in the path integral. In this sense the path integral has little flexibility as to modified time ordering operations. In some examples such as higher derivative theory, we have shown that the canonical commutation relations are readily recovered from the correlation functions defined by the path integral.

We analyzed some of the basic aspects of quantized theory which is non-local in the time variable on the basis of the path integral quantization. In general, the naive quantization in the sense of the interaction picture is not justified, but we showed that the space-time noncommutative theory is stable under higher order quantum corrections in a perturbative sense in sharp contrast to a naive theory non-local in time. Although we analyzed this issue for a simple scalar theory, we expect that the conclusion is valid for a more general class of field theories and thus this provides a basis for a perturbative analysis of space-time noncommutative theory.

In view of various time-ordering operations available in the operator formulation, we analyzed the recent proposal of the modified time ordering prescription[11, 12], which generally defines a unitary S-matrix for a theory non-local in time variable. This freedom of the modified time ordering is not available for the path integral, and thus specific to the operator formulation. We showed that the unitary S-matrix has certain advantages but at the time it has several disadvantages, and the perturbative positive energy condition and a smooth Wick rotation to Euclidean theory, which are ensured by the Feynman's $m^2-i\epsilon$ prescription, are spoiled.

Since a quantization scheme of space-time noncommutative theory satisfactory in every respect is not known at this moment, our conclusion is that the path integral quantization scheme with Feynman's $m^2 - i\epsilon$ prescription is attractive, which is simple in principle and allows a smooth definition of Euclidean theory indispensable for some non-perturbative analyses. The path integral formulation displays the difficulty of the space-time noncommutative theory as an absence of the unitary S-matrix.

As for the compelling motivation for studying the noncommutative space and time in fundamental physics, one may count the recent developments related to string theory such as in [25, 26, 27], for example, but the analysis of such concrete examples is beyond the scope of the present paper.

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